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EFFECTIVE PROPERTIES OF RANDOM COMPOSITES

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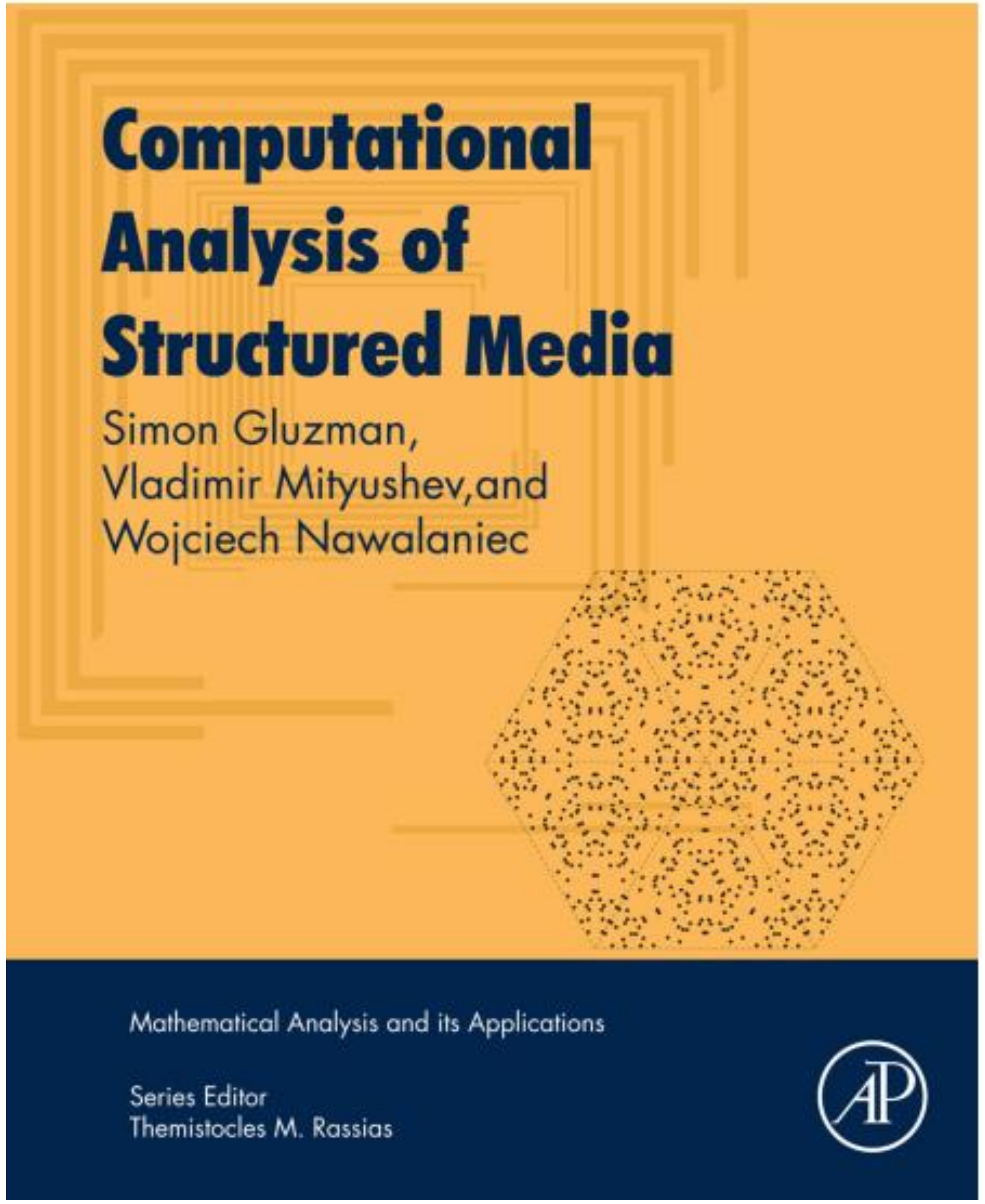
ABSTRACT

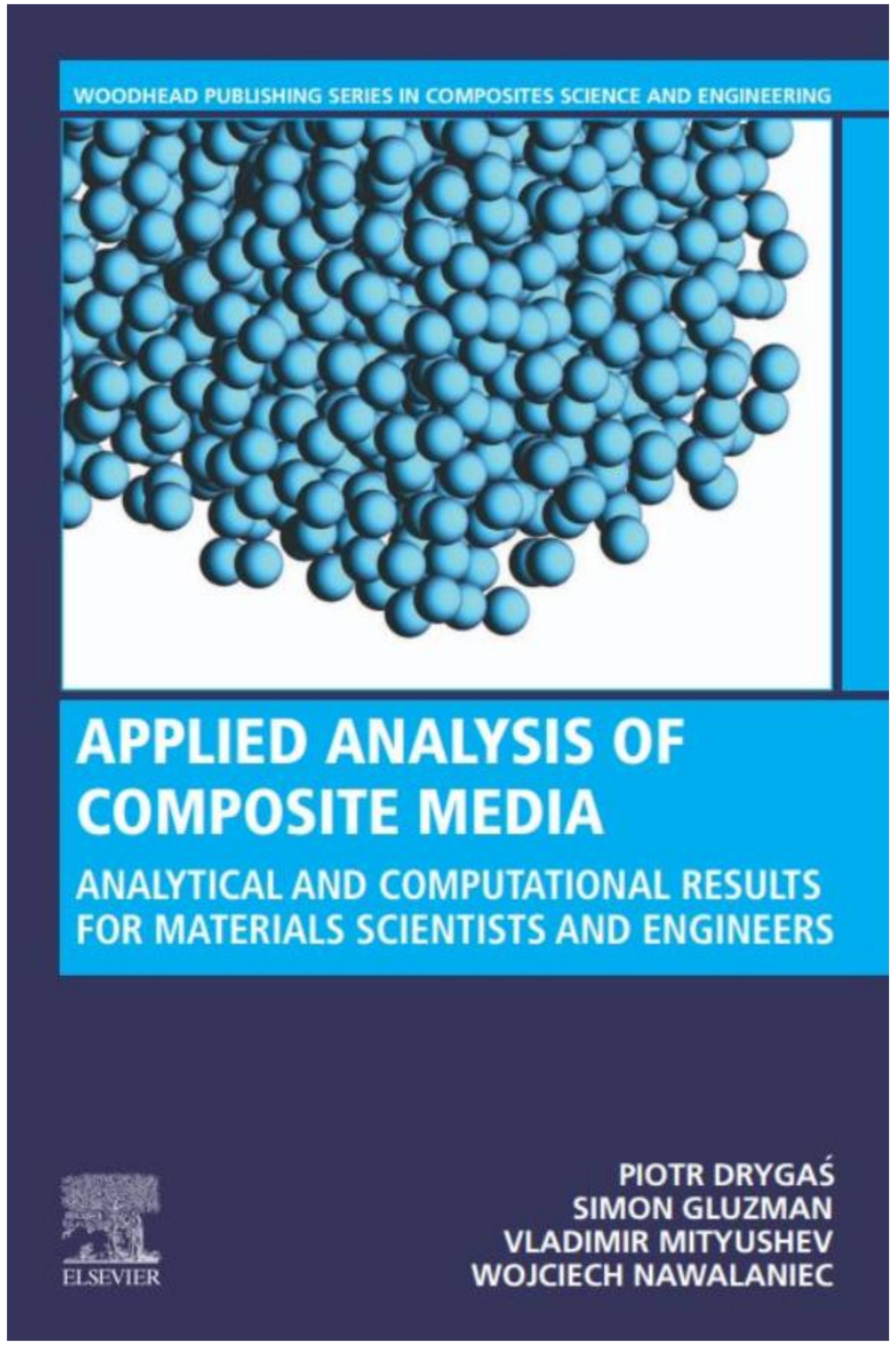
The talk is devoted to application of the Riemann-Hilbert and $\mathbb{R}$-linear problems for a multiply connected domain [VM 1996-2000] to the effective properties of 2D random composites [VM 2001-2021].

Various analytic formulas for random composites were deduced by means of self-consistent methods (effective medium approximation etc.) In many cases, a formula was derived having used physical arguments and manipulations with the further declaration without necessary analysis that this formula is universal or valid at least for a wide class of composites. It is demonstrated that self-consistent methods leads at most to the old formulas for a dilute composites. Hence, misleading self-consistent approaches should be discarded, and exact and approximate analytical formulas for the $\mathbb{R}$-linear problem have to be applied.

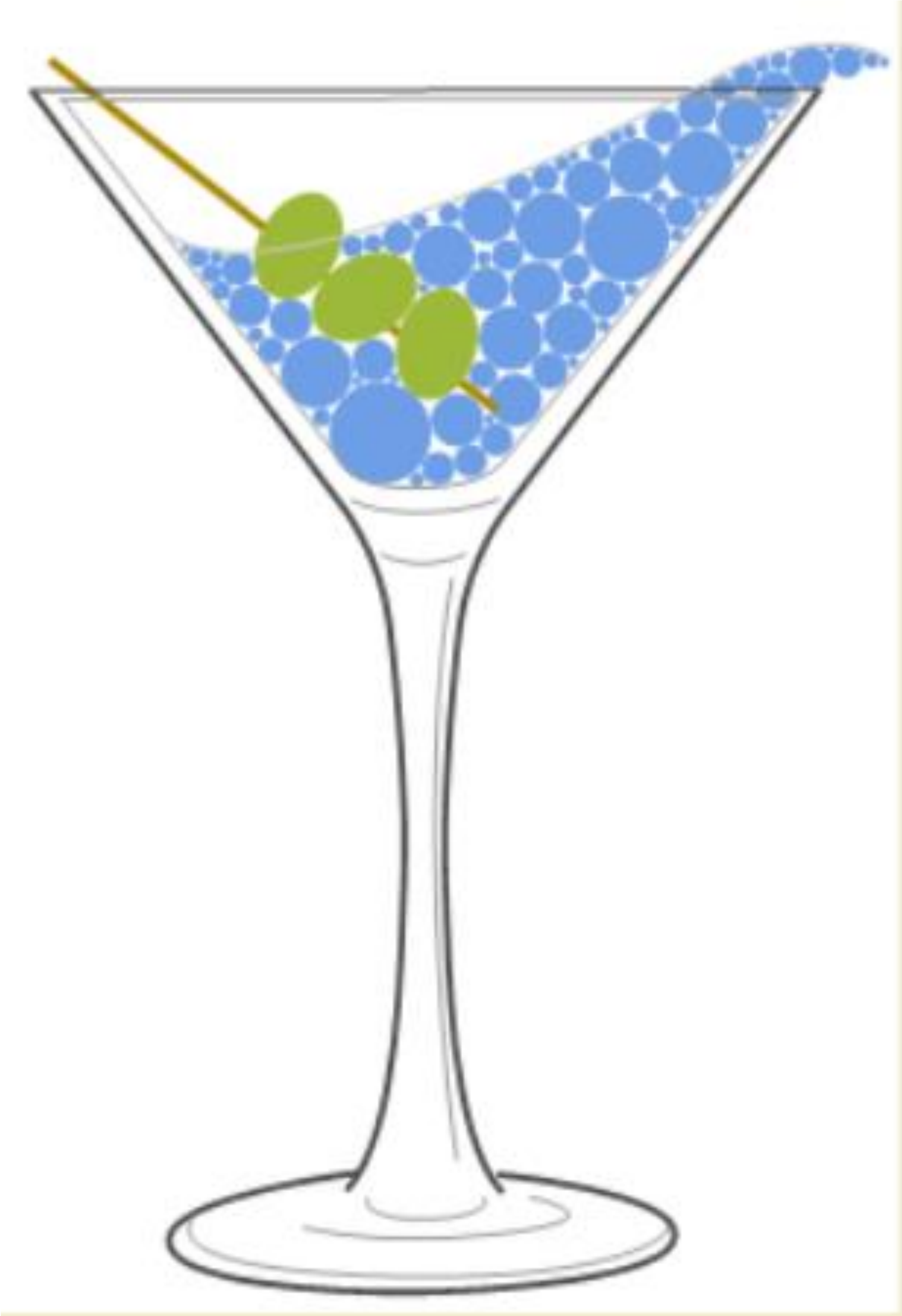
RECENT BOOKS 2018 (2D CONDUCTIVITY), 2020 (2D ELASTICITY & VISCOSITY, 3D CONDUCTIVITY)

https://www.amazon.com/Vladimir-V.-Mityushev/e/B001K8D332





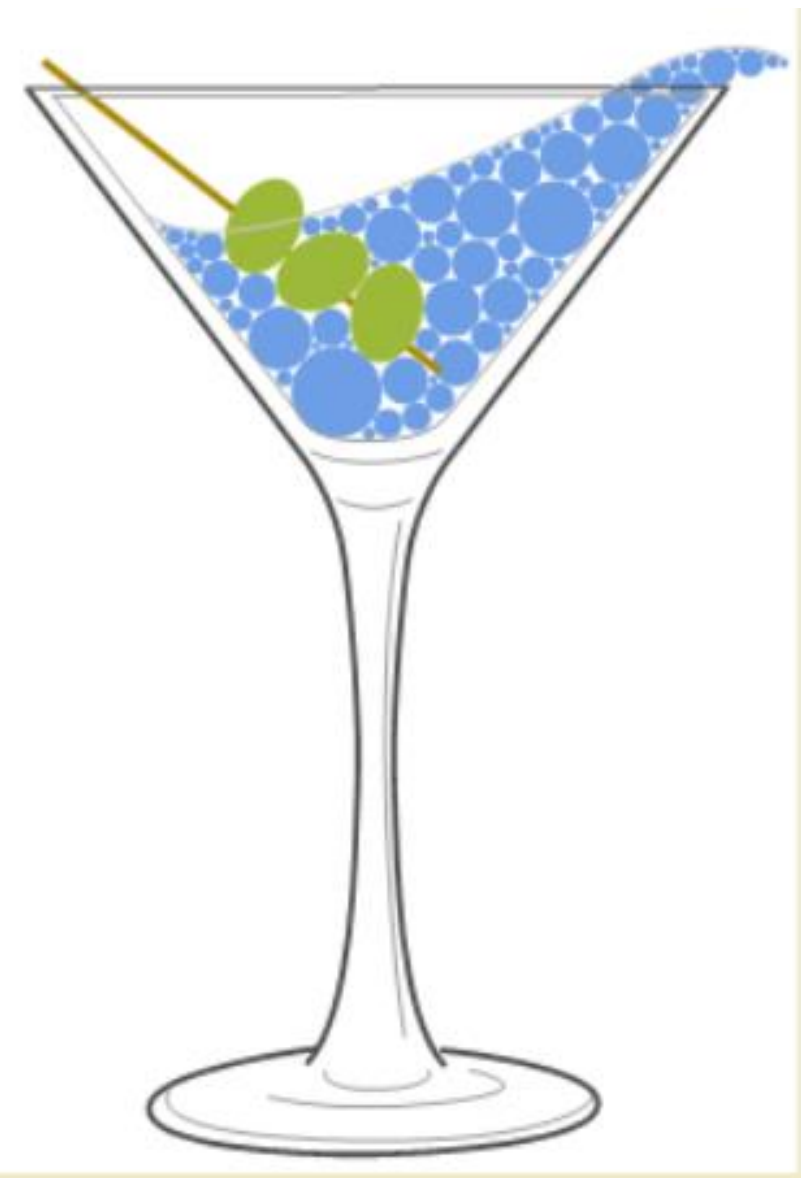
WHY DID JAMES BOND PREFER SHAKEN, NOT STIRRED MARTINI WITH ICE?

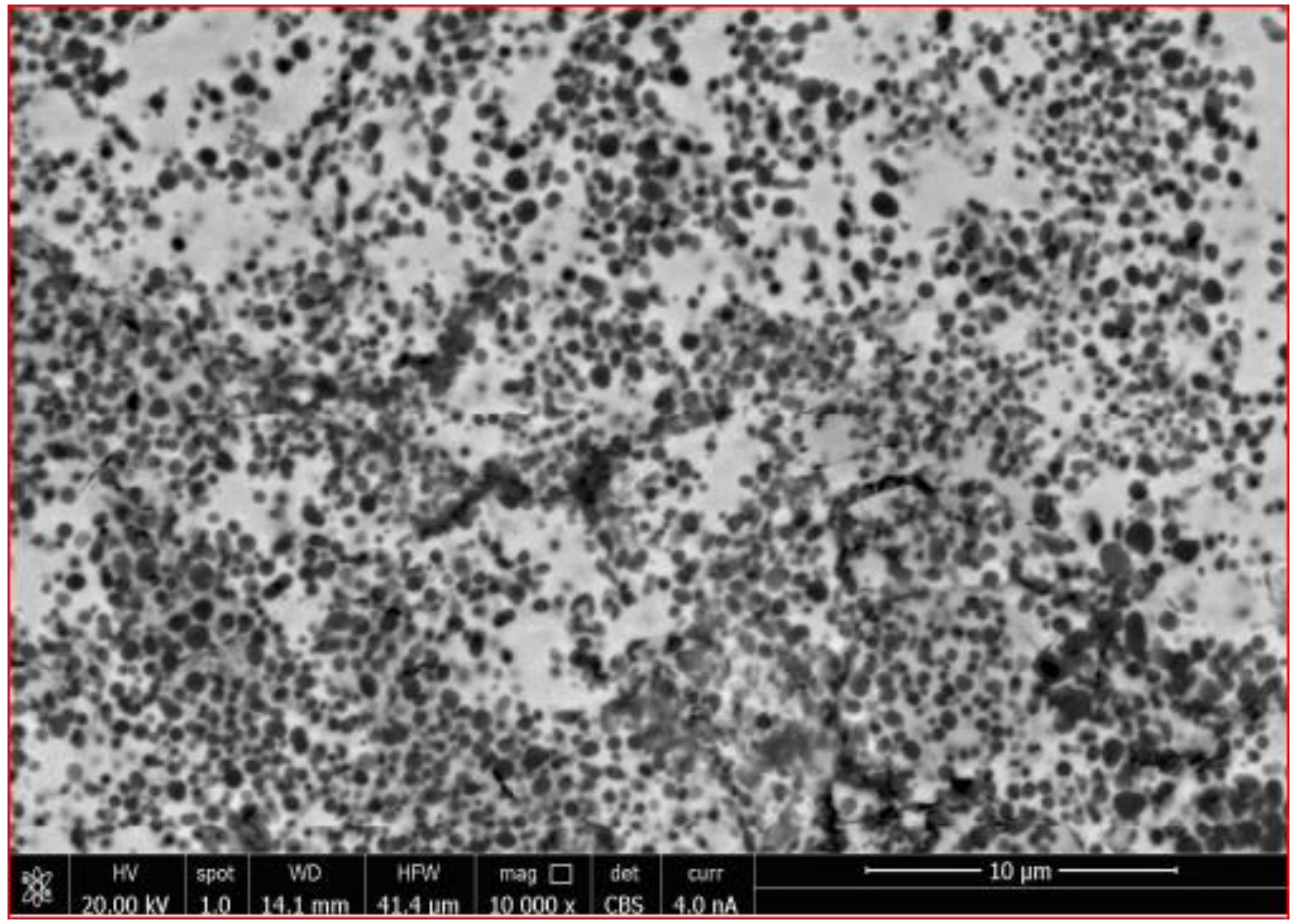


Why did James Bond prefer shaken,

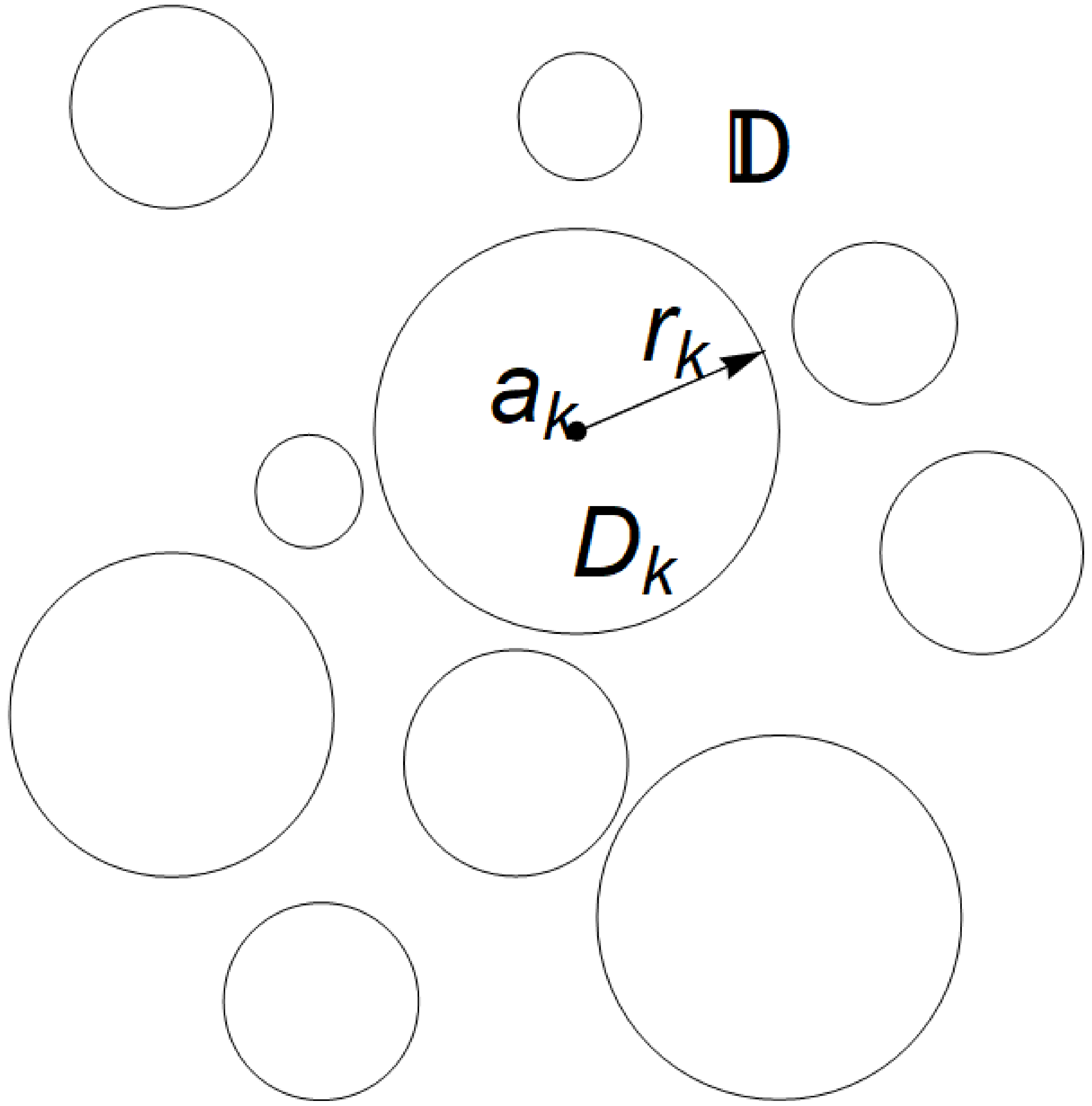
not stirred martini with ice?

2D STATIONARY PROBLEM





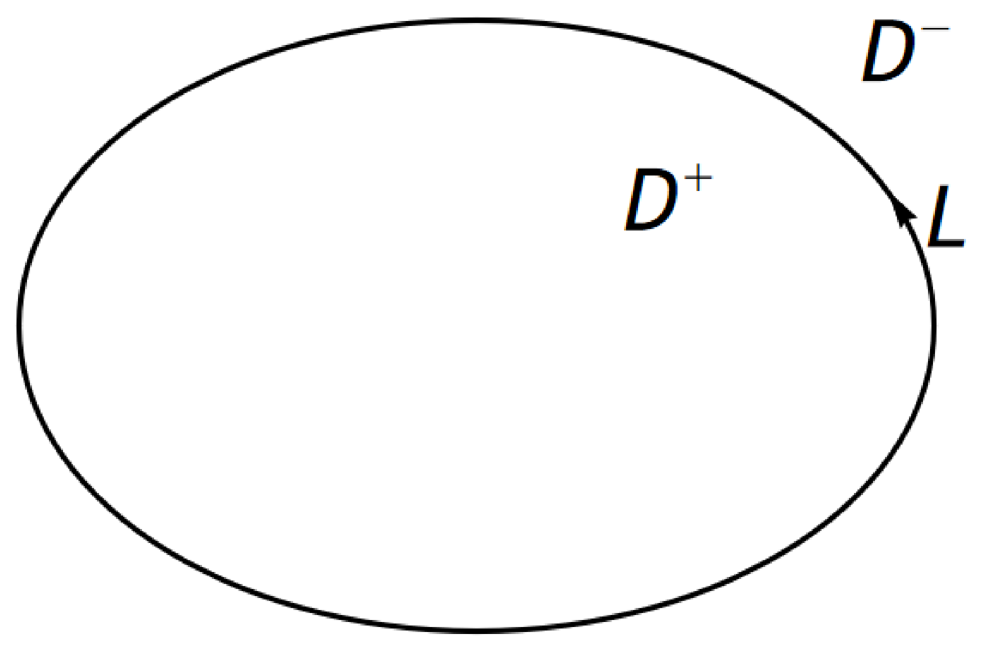
Microstructure of TiC–FeCr composite



The present talk concerns 2D stationary conductivity problems for disks. The unknown functions are analytic in the considered domains and Hölder continuous in their closures.

RIEMANN-HILBERT AND $\mathbb{R}$-LINEAR PROBLEMS

simply connected domain

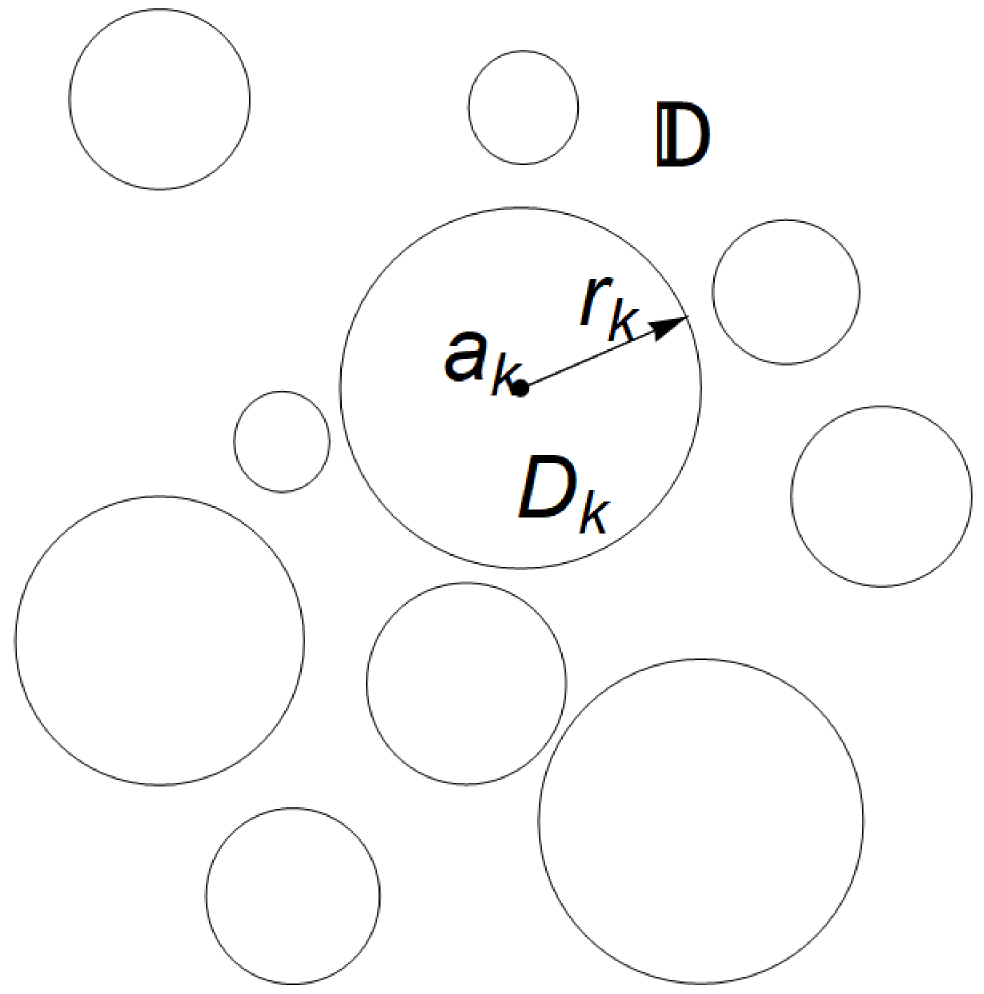


$$\phi^+(t) = a(t)\,\phi^-(t) \;+\; b(t)\,\overline{\phi^-(t)} + c(t)\,, \quad t \in L. \qquad (\mathbb{R})$$

$Z = a\,W + b\,\overline{W} + c$ $\mathbb{R}$-linear relation between Z and W

The Riemann-Hilbert (RH) problem is a particular case of the $\mathbb{R}$-linear problem when $|a(t)| = |b(t)|$ (I. Sabitov 1959/60, not published):

multiply connected domain



$$\overline{\lambda(t)}\,\phi^+(t) = \phi^-(t) \;-\; \overline{\phi^-(t)} + c(t) \;\Rightarrow\; \mathrm{Re}\,\overline{\lambda(t)}\,\phi^+(t) = \mathrm{Re}\,c(t) \qquad (\mathbb{RH})$$

In the case of multiply connected domain $D^+ = \cup_{k=1}^{n} D_k$.

FIRST WORKS ON ℝ-LINEAR PROBLEMS

N.I. Muskhelishvili: To the problem of torsion and bending of beams constituted from different materials, Izv. AN SSSR, (1932), N 7, 907-945

I.N. Vekua, A.K. Rukhadze: The problem of the torsion of circular cylinder reinforced by transversal circular beam.Izv. AN SSSR, 1933, n. 3, 373-386.

I.N. Vekua, A.K. Rukhadze: Torsion and transversal bending of the beam compounded by two materials restricted by confocal ellipces. Prikladnaya Matematika i Mechanika (Leningrad), 1933, 1, n. 2, 167-178;

(many thanks to Gia Giorgadze for literature)

G.M. Golusin: Solution of basic plane problems of mathetical physics for the case of Laplace equation and multiply connected domains bounded by circles (method of functional equations). Math. zb. 41:2 (1934), 246-276.

A.I. Markushevich: On a boundary value problem of analytic function theory. Uch. zapiski MGU 1 (1946), 100, 20-30. $\phi^+(t) = a(t)\,\phi^-(t) + b(t)\,\overline{\phi^-(t)} + c(t)\,, \quad t \in L.$ $(\mathbb{R})$

B. Bojarski: On generalized Hilbert boundary value problem, Soobsch. AN GruzSSR, 25 (1960), n. 4, 385-390;

extended to multiply connected domains [Bojarski & VM 2013] . (the case $|a(t)| > |b(t)|$)

L.G. Mikhailov: A new class of singular integral equations and its application to differential equations with singular coefficients, Dushanbe, 1963

RIEMANN-HILBERT AND ℝ-LINEAR PROBLEMS

The RH problem has been discussed in the classical books [Gakhov, Muskhelishvili, Vekua]. One can find the solution of RH problem in closed form for simple ($n = 1$), double connected domains [$n = 2$, see Bancuri] and in other special cases. Complete solution of the scalar problem RH problem was obtained in analytic form (VM 1996, 1998 with an extended presentation VM 2012).

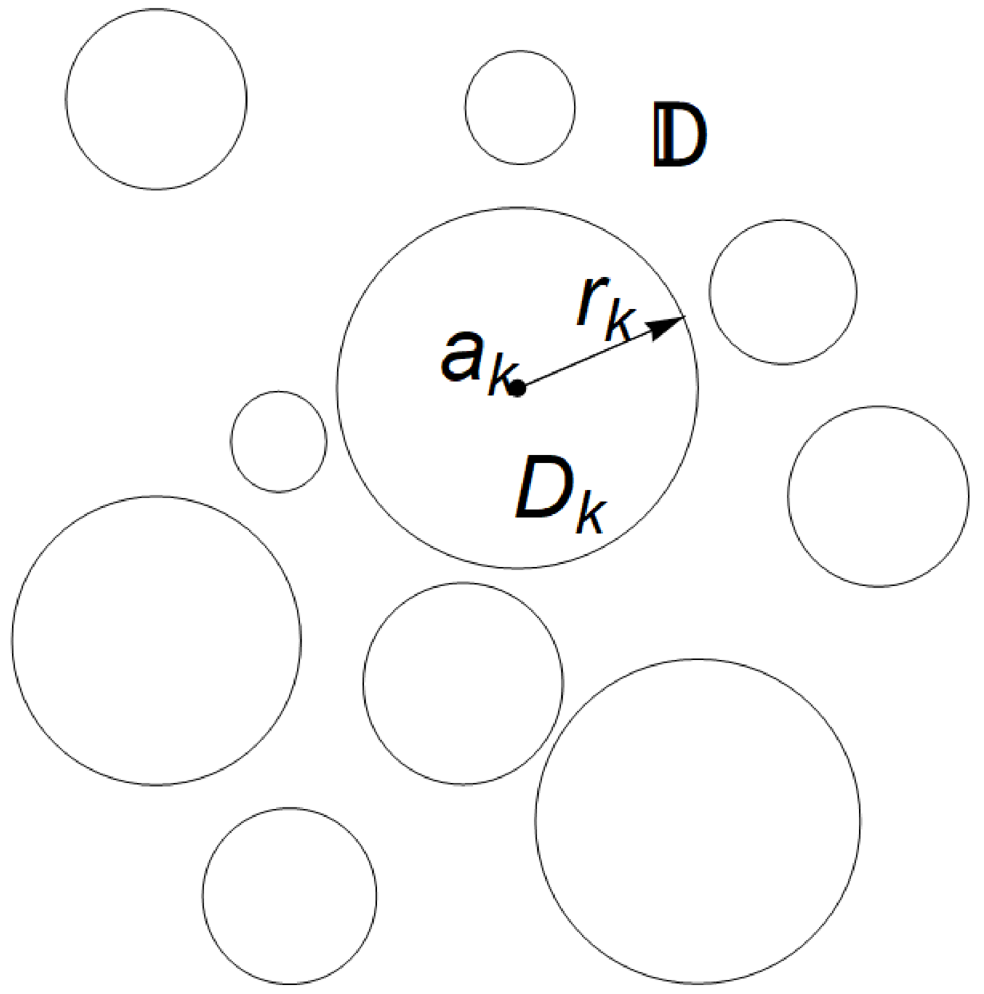
Scheme of solution:

i) to reduce the problem to a multiply connected circular domain by a conformal mapping;

ii) to apply the standard method of factorization to reduce to the blocks of problems

iii) to reduce the RH with constant coefficients to the ℝ-linear problem;

iv) to reduce the ℝ-linear problem to functional equations;



$$\mathrm{Re}\,\overline{\lambda_k}\phi(t) = g_k(t) \ (k = 1,2,\dots,n) \ \Leftrightarrow \ \phi(t) = \lambda_k\phi_k(t) - \lambda_k\overline{\phi_k(t)} + \lambda_k g_k(t) \Leftrightarrow$$

$$\lambda_k\phi_k(t) - \phi(t) = \lambda_k\overline{\phi_k(t)} - \lambda_k g_k(t) \Leftrightarrow \ \phi_k(z) = \frac{\overline{\lambda_k}\lambda_m}{2\pi i}\sum_{m=1}^{n}\int_{L_m}\frac{\overline{\phi_m(t)}}{t-z}dt + f(z), \quad z \in D_k$$

Sochotski's formulas

v) to solve the functional equations in terms of the Poincaré series.

POISSON FORMULA

Consider the exterior of the unit disk | z | > 1 in the complex plane. Let $z = r\, e^{i\,\theta}$

The function

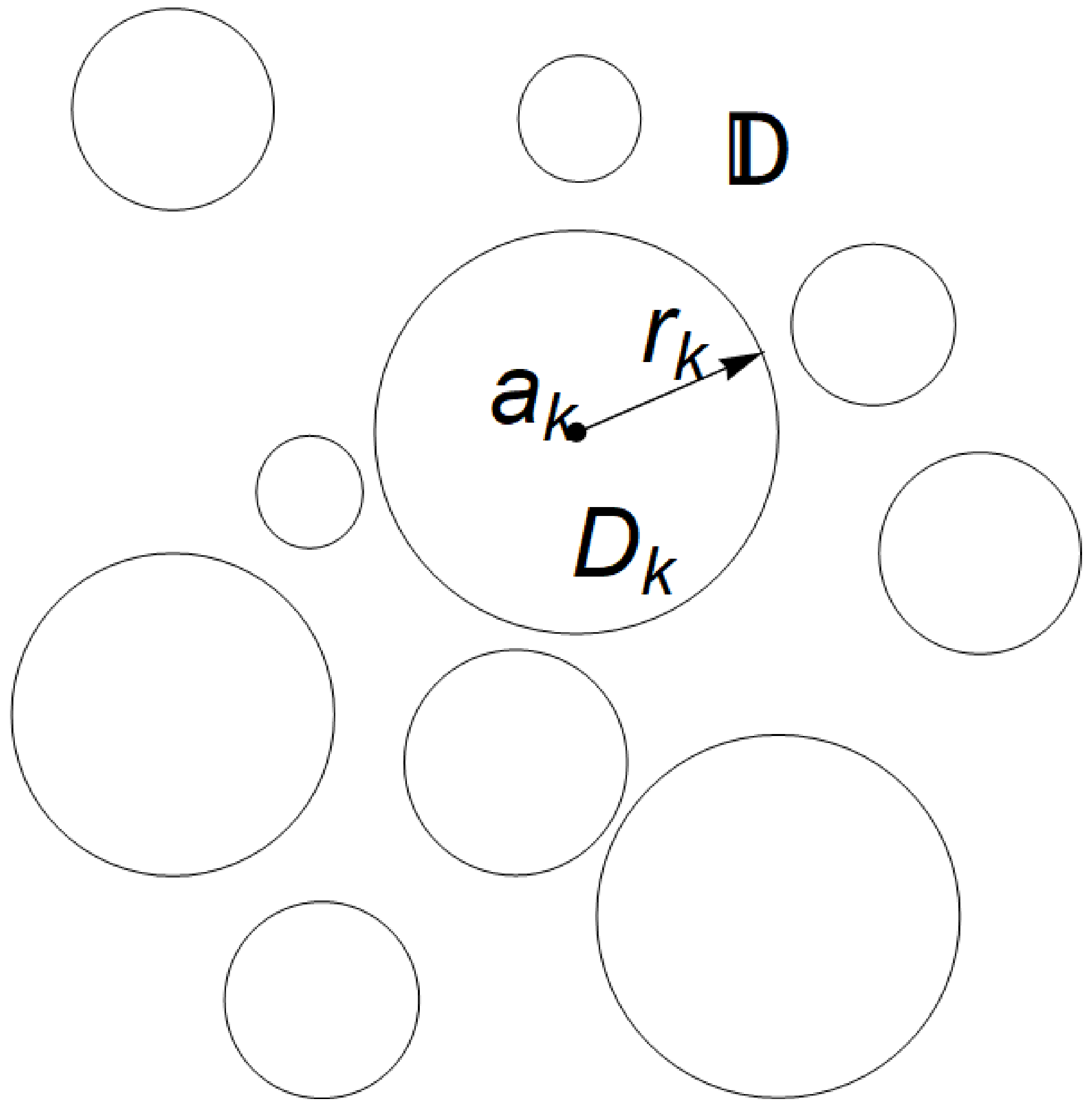
$$u(z;\{0,1\}) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{|z|^2-1}{|z e^{-i\theta}-1|^2} f(e^{i\theta})\, d\theta$$

solves the Dirichlet problem for the unit disk

$$u(z) = f(z), \qquad |z| = 1.$$

Extension to the multiply connected domains (VM 1996, 1998):

$$u(z;\{\{a_1,r_1\},\ \dots,\ \{a_n,r_n\}\}) = \sum_{k=1}^{n}\int_{-\pi}^{\pi} F_k(z,\theta)\, f_k(e^{i\theta})\, d\theta.$$



COMPLEX GREEN'S FUNCTION

Complex Green's function is represented in the form

$$M(z,\zeta) = M_0(z,\zeta) + \sum_{m=1}^{n} \alpha_m(\zeta) \ln(\zeta - a_m) - \ln(\zeta - z) + A(\zeta),$$

where $M_0(z,\zeta)$ is a single-valued analytic function of z in D (for any fixed ζ), $M_0(w,\zeta) = 0$;

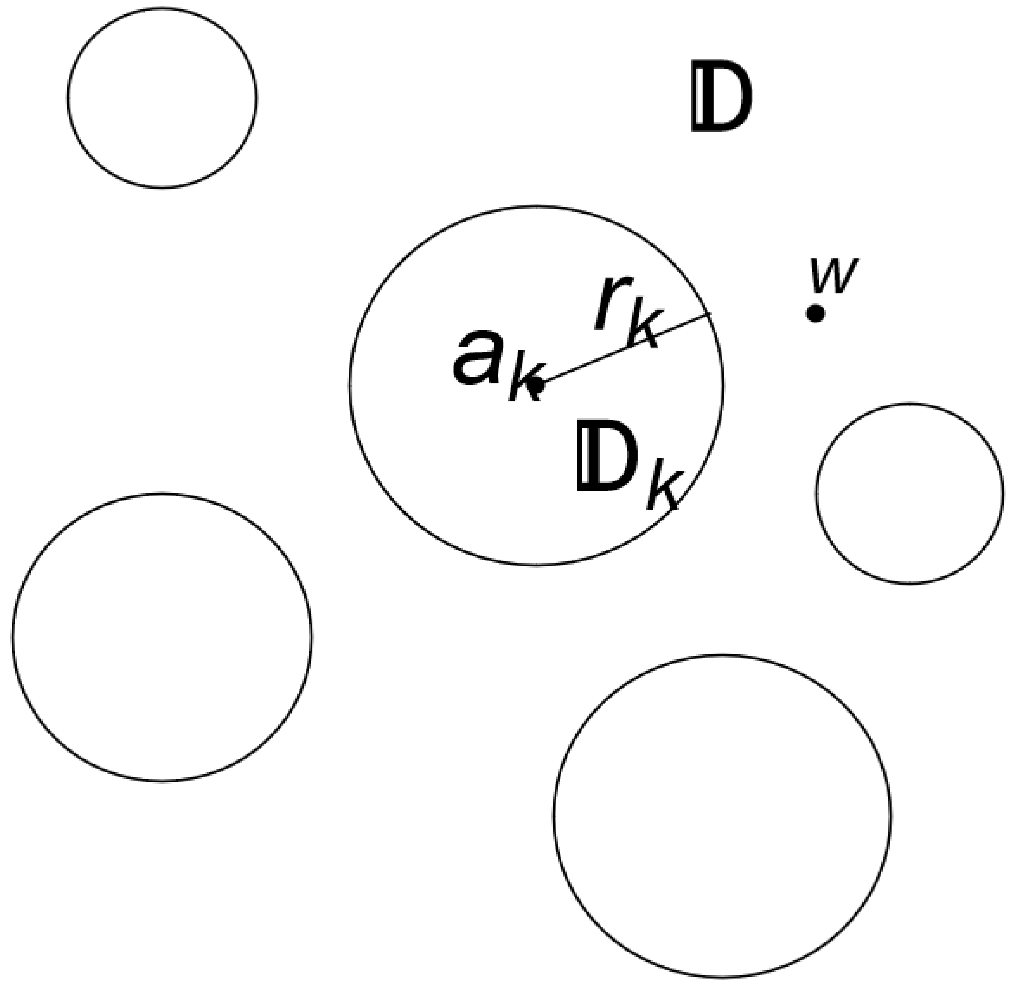
$\alpha_m(\zeta), A(\zeta)$ are unknown. The boundary value problem for $M_0(z,\zeta)$:

$$Re[\, M_0(z,\zeta) + \sum_{m=1}^{n} \alpha_m(\zeta) \ln(\zeta - a_m) - \ln(\zeta - z) + A(\zeta)] = 0, \quad |z - a_k| = r_k,\ k = 1,2,\dots,n.$$

It is reduced to the system of functional equations

$$\phi_k(z) = -\sum_{m\neq k}\left[\overline{\phi_m\left(z^*_{(m)}\right)} - \overline{\phi_m\left(w^*_{(m)}\right)}\right] - f(z), \quad |z - a_k| \leq r_k,\ k = 1,2,\dots,n.$$

where $z^*_{(m)} = \frac{r_m^2}{\overline{z-a_m}} + a_m$ is the inversion with respect to the circle $|t - a_m| = r_m$.



The composition of two inversions generate a Möbius transformation

$$\gamma_{j(z)} = \frac{a_j z + b_j}{c_j z + d_j} = \left(z^*_{(m)}\right)^*_{(k)}$$

COMPLEX GREEN'S FUNCTION

Application of successive approximations to the functional equations yields the uniformly convergent series

$$M_0(z,\zeta) = \sum_{m=1}^{n} \left(\alpha_m(\zeta)\ \ln \prod_{j=1, j\neq m}^{\infty} \psi_m^j(z)\right) + \ \ln \prod_{j=1}^{\infty} \omega_j(z),$$

$$\omega_j(z,\zeta) = \begin{cases} \dfrac{\zeta-\gamma_j(z)}{\zeta-\gamma_j(w)} & \text{if} \quad \gamma_j \in \mathcal{K}, \\ \overline{\dfrac{\overline{\zeta-\gamma_j(\bar{z})}}{\zeta-\gamma_j(\overline{w})}} & \text{if} \quad \text{if } \gamma_j \in \mathcal{F}. \end{cases} \qquad \psi_m^j(z) = \begin{cases} \dfrac{\gamma_j(z)-a_m}{\gamma_j(w)-a_m} & \text{if} \quad \gamma_j \in \mathcal{K}, \\ \overline{\dfrac{\overline{\gamma_j(\bar{z})-a_m}}{\gamma_j(\overline{w})-a_m}} & \text{if} \quad \text{if } \gamma_j \in \mathcal{F}. \end{cases}$$

$\mathcal{K}$ consists of even order inversions (the classic Schottky group),

$\mathcal{F}$ of odd order inversions

SCHWARZ'S OPERATOR

Theorem (Schwarz's operator for a circular multiply connected domain)

The RH problem

$$\mathrm{Re}\ \phi(t) = f(t), \qquad |t - a_k| = r_k \quad (k = 1,2,\ldots,n)$$

with single-valued $\phi(z)$ in *D* is solvable if and only if a system of *n* linear algebraic equations (Bojarskij's system 1958) is solvable. $\phi(z)$ can be written explicitly in terms of uniformly convergent series

$$\phi(z) = \frac{1}{2\pi \mathrm{i}} \sum_{k=1}^{n} \int_{\mathbb{T}_k} (f(\zeta) + c_k) \left\{ \sum_{j=2}^{\infty}{}'' \left[\frac{1}{\zeta - \gamma_j(w)} - \frac{1}{\zeta - \gamma_j(z)} \right] \right.$$
$$\left. + \left(\frac{r_k}{\zeta - a_k} \right)^2 \sum_{j=1}^{\infty}{}' \left[\frac{1}{\overline{\zeta - \gamma_j(\overline{z})}} - \frac{1}{\overline{\zeta - \gamma_j(\overline{w})}} \right] - \frac{1}{\zeta - z} \right\} \mathrm{d}\zeta$$
$$+ \frac{1}{2\pi \mathrm{i}} \sum_{k=1}^{n} \int_{\mathbb{T}_k} f(\zeta) \frac{\partial A}{\partial \nu}(\zeta) \mathrm{d}\sigma + i\varsigma\,.$$

Bojarskij's system:

$$\sum_{k=1}^{n} \int_{\mathbb{T}_k} (f(\zeta) + c_k) \frac{\partial \alpha_m}{\partial \nu}(\zeta) \mathrm{d}\sigma = 0\,, \quad m = 1,2,\ldots,n-1\,.$$

$$\alpha_s(z) = \sum_{m=1}^{n} A_m [\mathrm{Re}\ \psi_m(z) + \ln|z - a_m|] + A\,,$$

$$\psi_m(z) = \ln \left[\prod_{j \in \mathscr{K}_m}^{\infty} \psi_m^{(j)}(z) \right] \qquad \mathscr{K}_m = \left\{ z^*_{(k_p k_{p-1} \ldots k_1)} : k_p \neq m \right\}$$

POINCARÉ SERIES

Let $H(z)$ be a meromorphic function. The Poincaré series is associated with the group $\mathcal{K}$ of inversions with respect to $|t - a_k| = r_k$

$$\theta_{2q}(z) := \sum_{j=0}^{\infty} H(\gamma_j(z))(c_j z + d_j)^{-2q}$$

H. Poincaré (1883) proved the absolute convergence of $\theta_4(z)$ and just said

„Toujours dans le cas d'un groupe fuchsien, la serié

$$\sum mod(c_j z + d_j)^{-2}$$

n'est par convergent ".

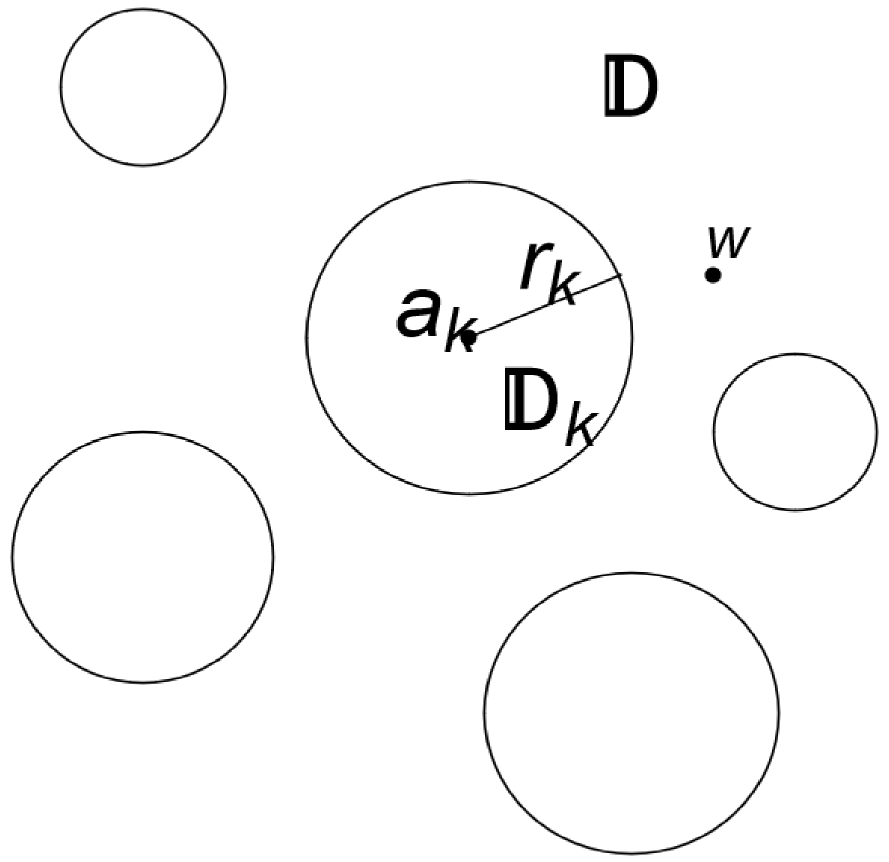
W. Burnside (1891) gave examples of convergent series for Schottky groups and studied their absolute convergence under some geometrical restrictions. In his study Burnside followed Poincaré's proof of the convergence of the θ_4-series.

Burnside wrote „I have endeavoured to show that, in the case of the first class of groups, this series is convergent, but at present I have not obtained a general proof. I shall offer two partial proofs of convergency; one of which applies only to the case of Fuchsian groups, and for that case in general, while the other will also apply to Kleinian groups, but only when certain relations of inequality are satisfied."

POINCARÉ SERIES

However, Myrberg (1916) gave an example of absolutely divergent series for a 64-connected domain. Beginning from Myrberg many mathematicians justified the absolute convergence of the Poincaré series under geometrical restrictions to the locations of the circles. Here, we present such a typical restriction expressed in terms of the separated parameter Δ introduced by Henrici

$$\Delta = \max_{k \neq m} \frac{r_k + r_m}{|a_k - a_m|} < \frac{1}{(n-1)^{1/4}}$$



Akaza (1964-1984) described domains of absolute convergence.

But the story had been finished by the following

Theorem (VM 1998).The Poincaré θ_2-series for any classical Schottky group converges uniformly in every compact subset of D/{limit points of $\mathcal{K}$}.

The RH problem for an arbitrary circular multiply connected domain is solved

in terms of the Poincaré θ_2-series and by its modifications.

The main reason of failure up to 1998: specialists in

Complex analysis like too much the infinite point.

$$\int_w^z \sum_{n=1}^{\infty} \frac{1}{(n-t)^2}\, dt = \sum_{n=1}^{\infty} \left(\frac{1}{n-z} - \frac{1}{n-w} \right)$$

APPLICATIONS TO OTHER PROBLEMS FOR A MULTIPLY CONNECTED DOMAIN

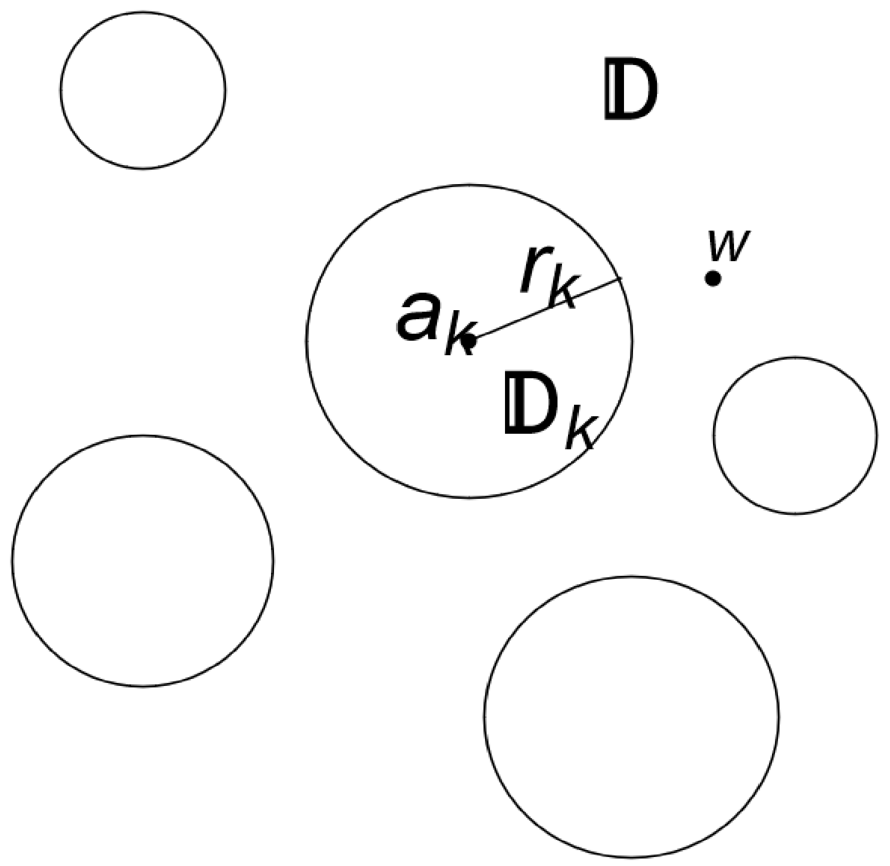
Bergman kernel (Moonja Jeong & VM, 2007),

Schottky-Klein prime function (VM 1998, 2012),

Jacobi inversion problem (VM 2012),

Schwarz-Christoffel integral (VM 2011-2012)

$$\mathrm{Re}[(t-a_k)\psi(t)] = -1, \qquad |t-a_k| = r_k,\ k=1,2,\dots,n$$



$$f : \mathbb{D} \longrightarrow P$$

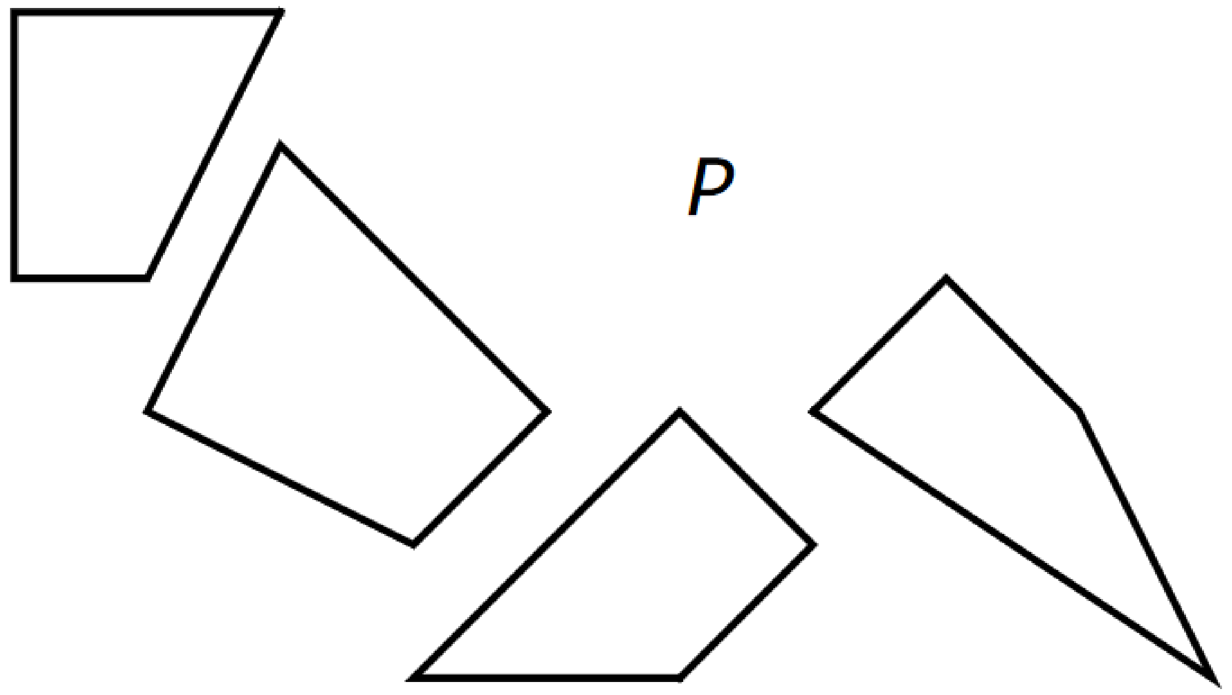
$$f(z) = \int^z \exp(\omega(\zeta))\, d\zeta$$

$$\exp(\omega(z)) = \prod_{m=1}^{n} \prod_{j=1}^{M_m} \left\{ \left(\frac{z_{\ell m} - z}{z_{\ell m} - w} \right)^{\beta_{\ell m}/2} \left[\prod_{k=1}^{n} \left(\overline{\frac{z_{\ell m} - z^*_{(k)}}{z_{\ell m} - w^*_{(k)}}} \right)^{\beta_{\ell m}/2} \right] \right.$$

$$\left. \times \left[\prod_{k=1}^{n} \prod_{k_1 \neq k} \left(\frac{z_{\ell m} - z^*_{(k_1 k)}}{z_{\ell m} - w^*_{(k_1 k)}} \right)^{\beta_{\ell m}/2} \right] \cdots \right\}$$

$$\times \left(\prod_{k=1}^{n} \frac{a_k - w}{a_k - z} \right) \left(\prod_{k=1}^{n} \prod_{k_1 \neq k} \overline{\frac{a_{k_1} - w^*_{(k)}}{a_{k_1} - z^*_{(k)}}} \right)$$

$$\times \left(\prod^{n} \prod \prod \frac{a_{k_2} - w^*_{(k_1 k)}}{a_{k_2} - z^*_{(k_1 k)}} \right) \cdots .$$



$\beta_{lm} = \alpha_{lm} - 1$ angle

z_{lm} vertex

$\mathbb{R}$ - LINEAR PROBLEM FOR A MULTIPLY CONNECTED DOMAIN

Let the inclusions D_k has the conductivity $\sigma^+ = \sigma_k$ $(k = 1, 2, \ldots, n)$ and the conductivity of D be normalized to unity as $\sigma^- = 1$. Such a normalization does not limit the generality of the problem. The coefficients σ_k then become dimensionless and are considered as the ratios of the conductivities of the kth inclusion to the conductivity of matrix. In these designations, (2.2.33) becomes

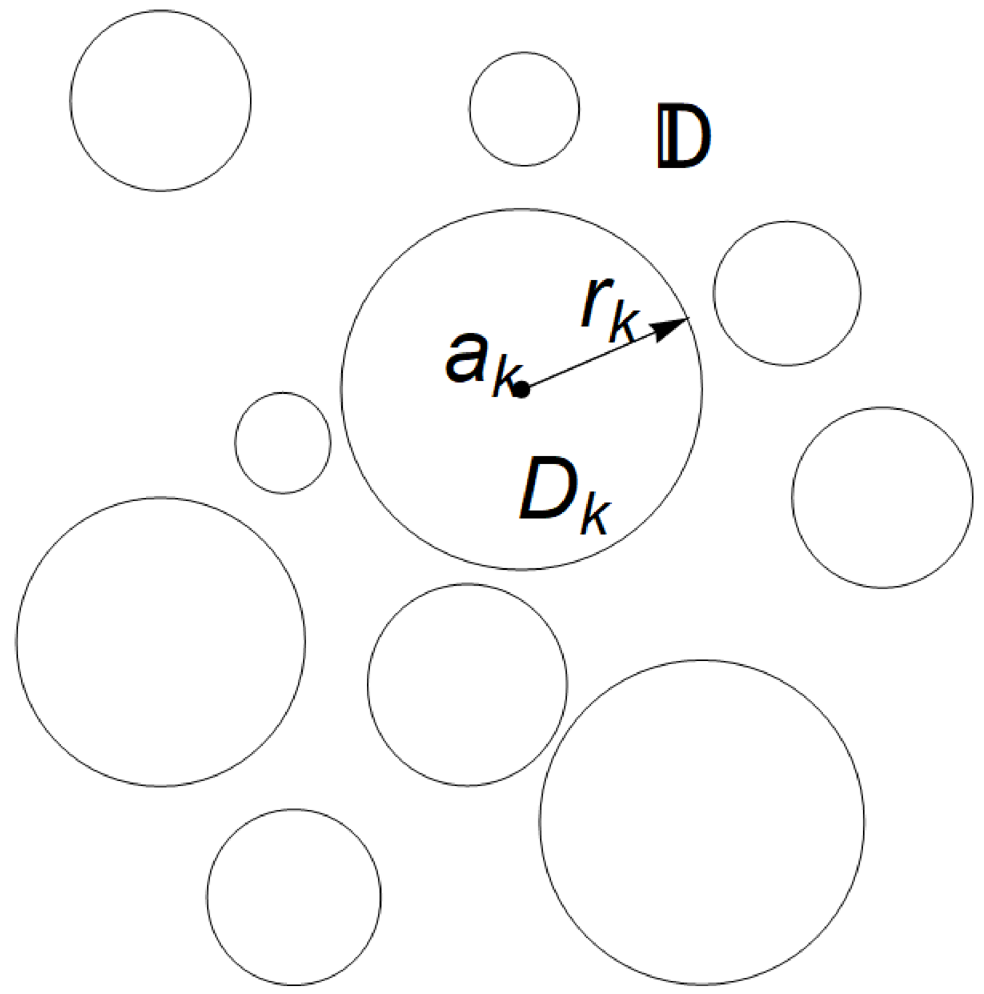
$$\boxed{u(t) = u_k(t), \quad \frac{\partial u}{\partial \mathbf{n}}(t) = \sigma_k \frac{\partial u_k}{\partial \mathbf{n}}(t), \quad t \in L_k \quad (k = 1, 2, \ldots, n).} \tag{2.2.37}$$

The subscript k pertains to the inclusions.

We now reduce (2.2.37) to an $\mathbb{R}$-linear problem. To this end, introduce the complex potentials $\varphi(z)$ and $\varphi_k(z)$ analytic (meromorphic) in D and D_k, respectively. The harmonic and analytic functions are related by the equalities

$$u(z) + iv(z) = \varphi(z), \quad z \in D, \tag{2.2.38}$$

$$u_k(z) + iv_k(z) = \frac{2}{\sigma_k + 1}\varphi_k(z), \quad z \in D_k \ (k = 1, 2, \ldots, n), \tag{2.2.39}$$



$\mathbb{R}$-linear problem $\boxed{\varphi(t) = \varphi_k(t) - \rho_k \overline{\varphi_k(t)}}$

Contrast parameter $\boxed{\rho_k = \dfrac{\sigma_k - 1}{\sigma_k + 1}}$

Complex flux $\boxed{\psi_k(z) \equiv \varphi_k'(z)}$

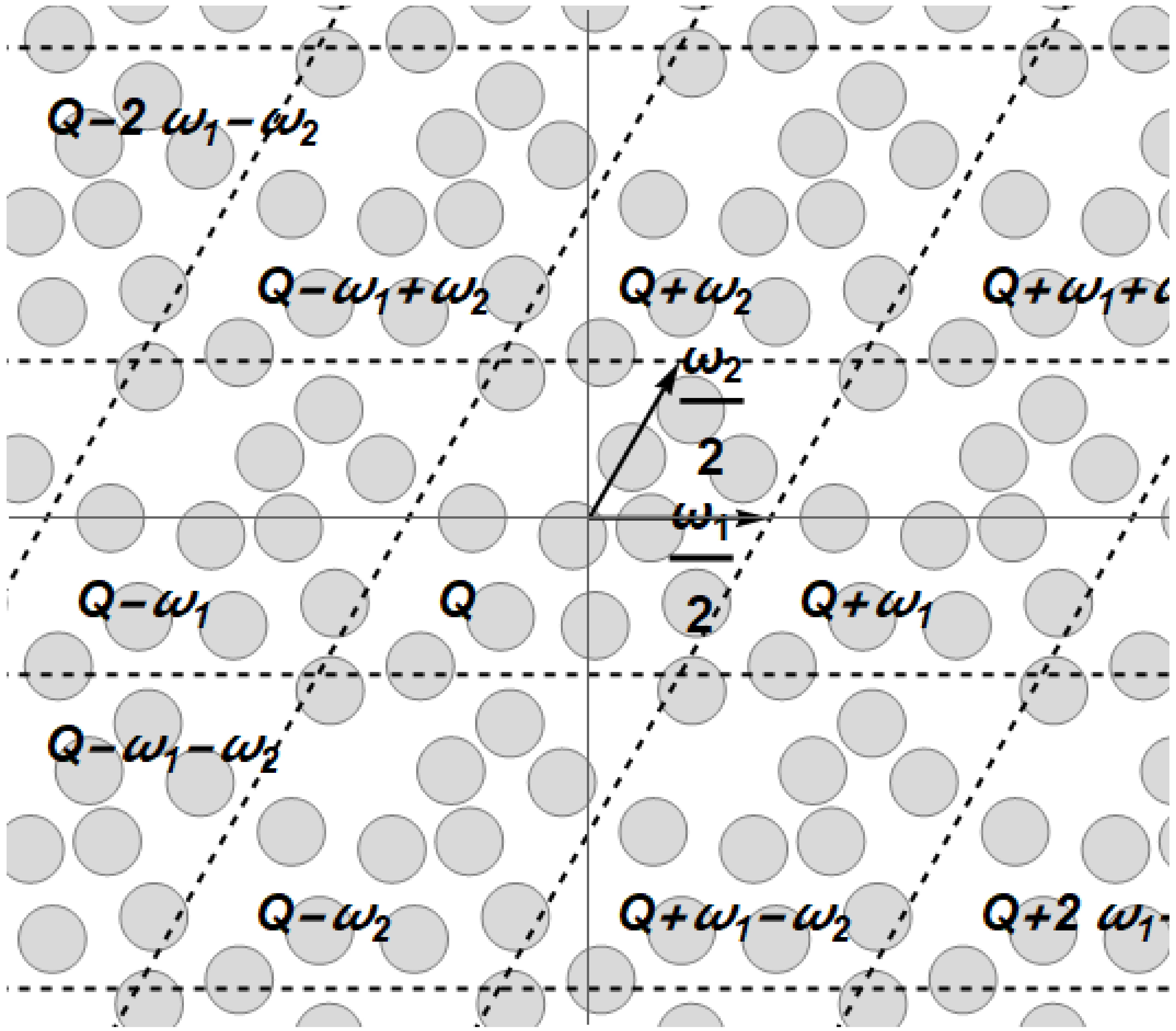
RIEMAN-HILBERT AND $\mathbb{R}$ -LINEAR PROBLEMS FOR DOUBLY PERIODIC DOMAIN (TORUS)

The $\mathbb{R}$- linear problem is reduced to the functional equations for $\psi_k(z) = \phi'_k(z)$

$$\psi_m(z) = \sum_{k=1}^{n} \rho \sum_{m_1, m_2}{}' \left(\frac{r_k}{z-a_k-m_1-i\,m_2}\right)^2 \overline{\psi_k\left(\frac{r_k}{\overline{z-a_k-m_1-i\,m_2}} + a_k\right)} + 1$$

$$|z-a_m| \le r_m, \quad m = 1, 2, \ldots, n.$$

Theorem 1. Let $|\rho| \leqslant 1$. The system of functional equations has a unique solution in a Banach space. This solution can be found by the method of successive approximations.



$$\sigma_{11} - i\sigma_{12} = 1 + 2\rho \sum_{k=1}^{N} \pi r_k^2 \psi_k(a_k).$$

In the case of equal radii, formula (3.2.43) becomes

$$\sigma_{11} - i\sigma_{12} = 1 + 2\rho f \frac{1}{N} \sum_{k=1}^{N} \psi_k(a_k),$$

where $f = N\pi r^2$ denotes the concentration of inclusions.

Efective conductivity tensor

$$\lambda_e = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$$

RIEMAN-HILBERT AND $\mathbb{R}$ -LINEAR PROBLEMS FOR DOUBLY PERIODIC DOMAIN (TORUS)

Let $E_m(z)$ denote the Eisenstein function.

Introduce the structural sums

$$S_2 = e_2 = \frac{1}{N^2}\sum_{k_0=1}^{N}\sum_{k_1=1}^{N} E_2(a_{k_0} - a_{k_1}),$$

$$e_{22} = \frac{1}{N^3}\sum_{k_0=1}^{N}\sum_{k_1=1}^{N}\sum_{k_2=1}^{N} E_2(a_{k_0} - a_{k_1})\overline{E_2(a_{k_1} - a_{k_2})}.$$

General structural sum

$$e_{m_1,\ldots,m_q} = \frac{1}{N^{1+\frac{1}{2}(m_1+\cdots+m_q)}}\sum_{k_0,k_1,\ldots,k_n} E_{m_1}(a_{k_0} - a_{k_1})\overline{E_{m_2}(a_{k_1} - a_{k_2})}\ldots \mathbf{C}^{q+1}E_{m_q}(a_{k_{q-1}} - a_{k_q})$$

For macroscopically isotropic composites $e_2 = \pi$

DECOMPOSITION SERIES FOR THE EFFECTIVE CONDUCTIVITY (PHYSICAL CONSTANTS, GEOMETRY, CONCENTRATION):

$$\lambda_e = 1 + 2\rho f + 2\rho \sum_{p=2}^{\infty} A[[p]]\, f^p$$

$$A[[2]] = \frac{\rho}{\pi} \frac{1}{n^2} \sum_{k_0,k_1}^{n} E_2\,(a_{k_0} - a_{k_1}),$$

$$A[[3]] = \frac{\rho^2}{\pi^2 n^3} \sum_{k_0,k_1,k_2}^{n} E_2\,(a_{k_0} - a_{k_1})\, \overline{E_2\,(a_{k_1} - a_{k_2})}$$

$$A[[4]] = \frac{1}{\pi^3 n^4} \left(-2\rho^2 \sum_{k_0,k_1,k_2}^{n} E_3\,(a_{k_0} - a_{k_1})\, \overline{E_3\,(a_{k_1} - a_{k_2})} + \rho^3 \sum_{k_0,k_1,k_2,k_3}^{n} E_2\,(a_{k_0} - a_{k_1})\, \overline{E_2\,(a_{k_1} - a_{k_2})}\, E_2\,(a_{k_2} - a_{k_3}) \right)$$

$$A[[5]] = \frac{1}{\pi^4} \left(6\rho^2 e_{4,4} - 2\rho^3\,(e_{3,3,2} + e_{2,3,3}) + \rho^4 e_{2,2,2,2}\right)$$

$$A[[6]] = \frac{1}{\pi^5} \left(-24\rho^2 e_{5,5} + 6\rho^3\,(e_{4,4,2} + e_{3,4,3} + e_{2,4,4}) - 2\rho^4\,(e_{3,3,2,2} + e_{2,3,3,2} + e_{2,2,3,3}) + \rho^5 e_{2,2,2,2,2}\right)$$

$$A[[7]] = \frac{1}{\pi^6} \big(120\rho^2 e_{6,6} - 24\rho^3\,(e_{2,5,5} + e_{3,5,4} + e_{4,5,3} + e_{5,5,2}) +$$
$$46\rho^4\,(e_{2,2,4,4} + e_{2,3,4,3} + e_{3,3,3,3} + e_{2,4,4,2} + e_{3,4,3,2} + e_{4,4,2,2}) - 24\rho^5\,(e_{2,2,2,3,3} + e_{2,2,3,3,2} + e_{2,3,3,2,2} + e_{3,3,2,2,2}) + \rho^6 e_{2,2,2,2,2,2}\big)$$

Percolation. Resummation techniques,

S. Gluzman, VM, W. Nawalaniec (2014)

The main idea consists in the asymptotic study of the function:

$$f(x) = (x - x_c)^{-s} g(x),$$

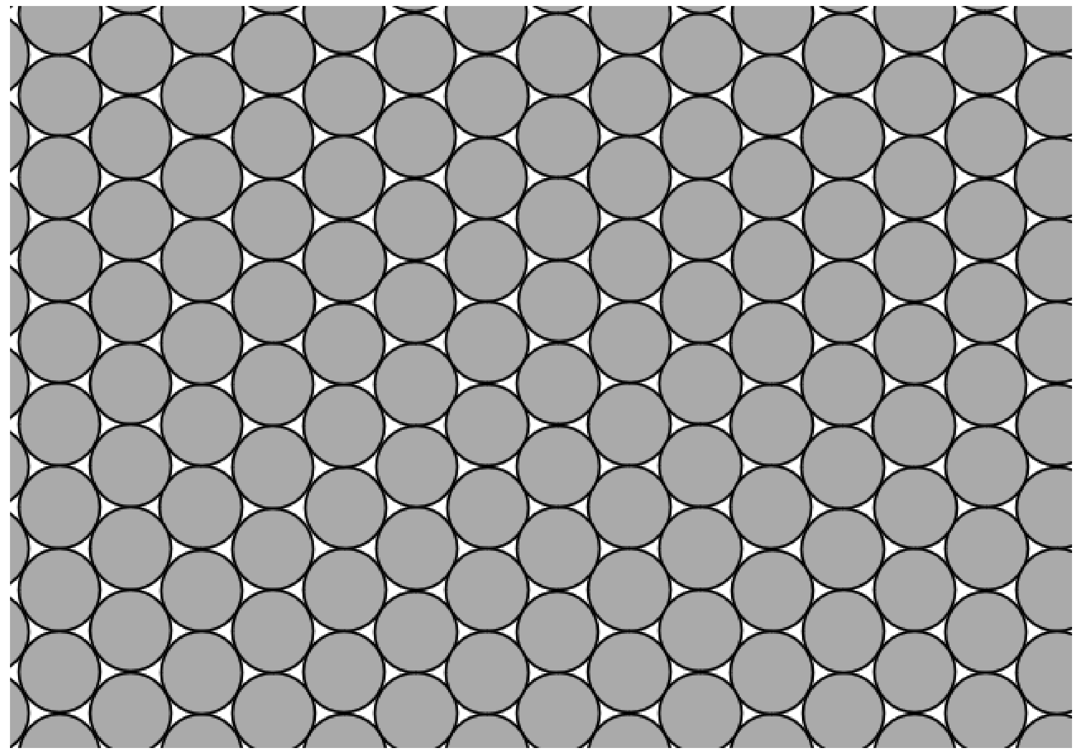
where s is unknown and x_c is known. We have

$$\ln f(x) = -s \ln(x - x_c) + \ln g(x) \quad \Leftrightarrow$$

$$\ln f(x) = s \left[\frac{x}{x_c} + \frac{1}{2}\left(\frac{x}{x_c}\right)^2 + \frac{1}{3}\left(\frac{x}{x_c}\right)^3 \right] + \ln g(x)$$

Remark. It is related to the Padé approximations which reveals the critical concentration x_c.

Hexagonal array for $\rho = 1$ (perfect conductor)



$$\text{Hex}[\nu] = 1 + 2\nu + 2\nu^2 + 2\nu^3 + 2\nu^4 + 2\nu^5 + 2\nu^6 + 2.15084\nu^7 + 2.30169\nu^8 + 2.45253\nu^9 + 2.60338\nu^{10} + 2.75422\nu^{11} + 2.90507\nu^{12} + 3.06744\nu^{13} + 3.24119\nu^{14} + 3.42632\nu^{15} + 3.62283\nu^{16} + 3.83071\nu^{17} + 4.04997\nu^{18} + 4.44142\nu^{19} + 4.84599\nu^{20} + 5.26454\nu^{21} + 5.69792\nu^{22} + 6.14699\nu^{23} + 6.61261\nu^{24} + 7.13504\nu^{25} + 7.70007\nu^{26}$$

$$\text{Hex}[\nu] = \left(\frac{36.1415}{\sqrt{\frac{\pi}{\sqrt{12}} - \nu}} + 15.991\sqrt{\frac{\pi}{\sqrt{12}} - \nu} - 45.685 + 2.462\nu \right) \frac{H_1[\nu]}{H_2[\nu]}$$

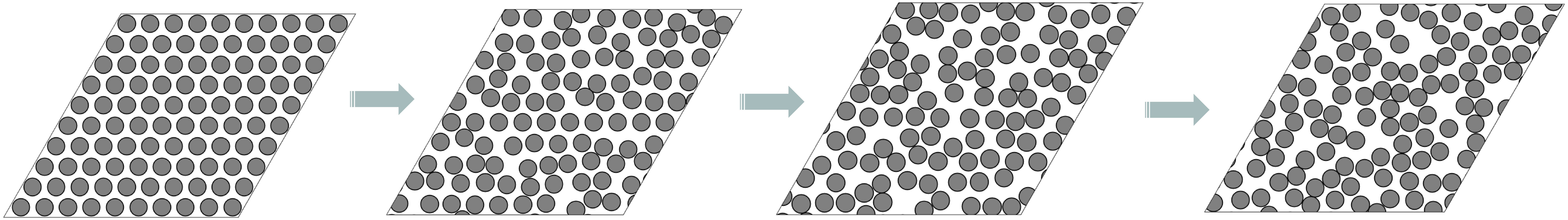
$$H_1[\nu] = \nu^7 + 0.063549\nu^6 + 0.625622\nu^5 + 0.65353\nu^4 + 0.627888\nu^3 - 5.18977\nu^2 + 1.377\nu + 6.94019,$$

$$H_2[\nu] = \nu^7 + 1.80866\nu^6 + 6.03947\nu^5 + 5.80087\nu^4 + 2.17086\nu^3 - 38.8956\nu^2 + 10.32\nu + 52.014$$

Here, $v = \pi r^2$ denotes the concentration of disks per unit cell

Non-overlapping uniform distribution

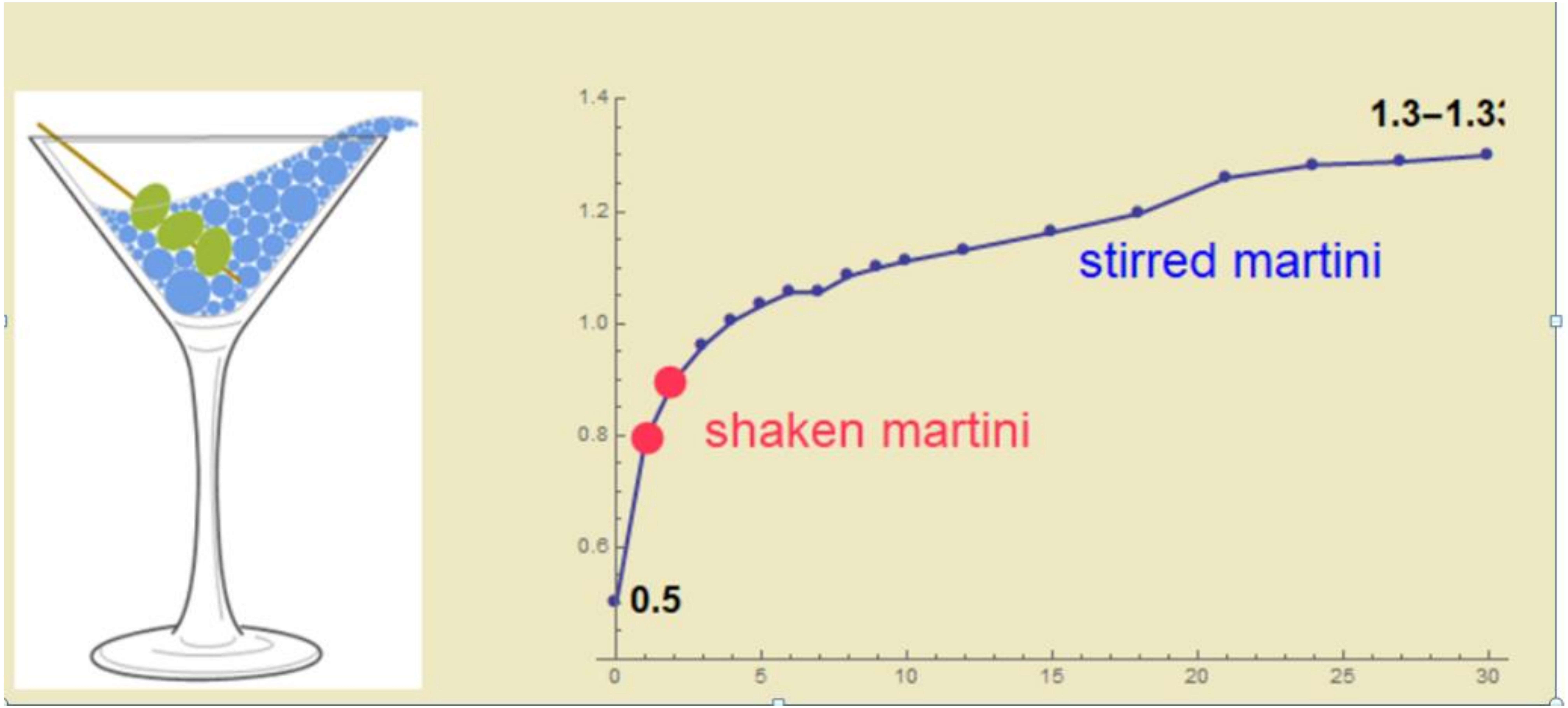
Random walks by 1000 Monte Carlo computational experiments



$$\mathrm{Random}[\nu] = 1 + 2\nu + 2\nu^2 + 5.00392\,\nu^3 + 6.3495\,\nu^4 + 0.0000186711\,\nu^9 + 9.57157\times10^{-10}\,\nu^{10} + 0.0570669\,\nu^{14} + 27.2148\,\nu^{15} + 7.06377\,\nu^{16} + 1.63666\times10^{-6}\,\nu^{17}$$

$$\lambda_e = 0.811521\left(\frac{(0.896003+v)\times(0.416762+0.114576\,v+v^2)}{(0.9069-v)\times(0.352057+0.34982\,v+v^2)}\right)^{1.3}$$

CRITICAL POWER



$\text{Fit}[t] = 0.5 + 0.82\, t^{\frac{1}{4}}$, where t is time of stirring (steps of random walks)

SELF CONSISTENT METHODS ETC.

Theorem (VM & Rylko, 2013)

Maxwell's approach and other self-consistent methods (SCMs) can give results valid only to $O((\rho\nu)^2)$ for isotropic composites.

Clausius (1879)–Mossotti (1850) or Maxwell (1873) approximation

$$\boxed{\lambda_e = 1 + 2\rho\nu + 2\rho^2\nu^2} + O(|\rho\nu|^3) = \frac{1+\rho\nu}{1-\rho\nu} + O(|\rho\nu|^3)$$

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There is a lot of *formulas* for the effective conductivity called *models*. See, for instance, Bruggeman (1935), Mori, Tanaka (1973) (*more than 8000 citations*) and reviews Karol Pietrak, Wisniewski (2015); Sevostianov, Mogilevskaya, Kushch (2019) based on the misleading methodology and including wrong formulas.

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Alchemy region

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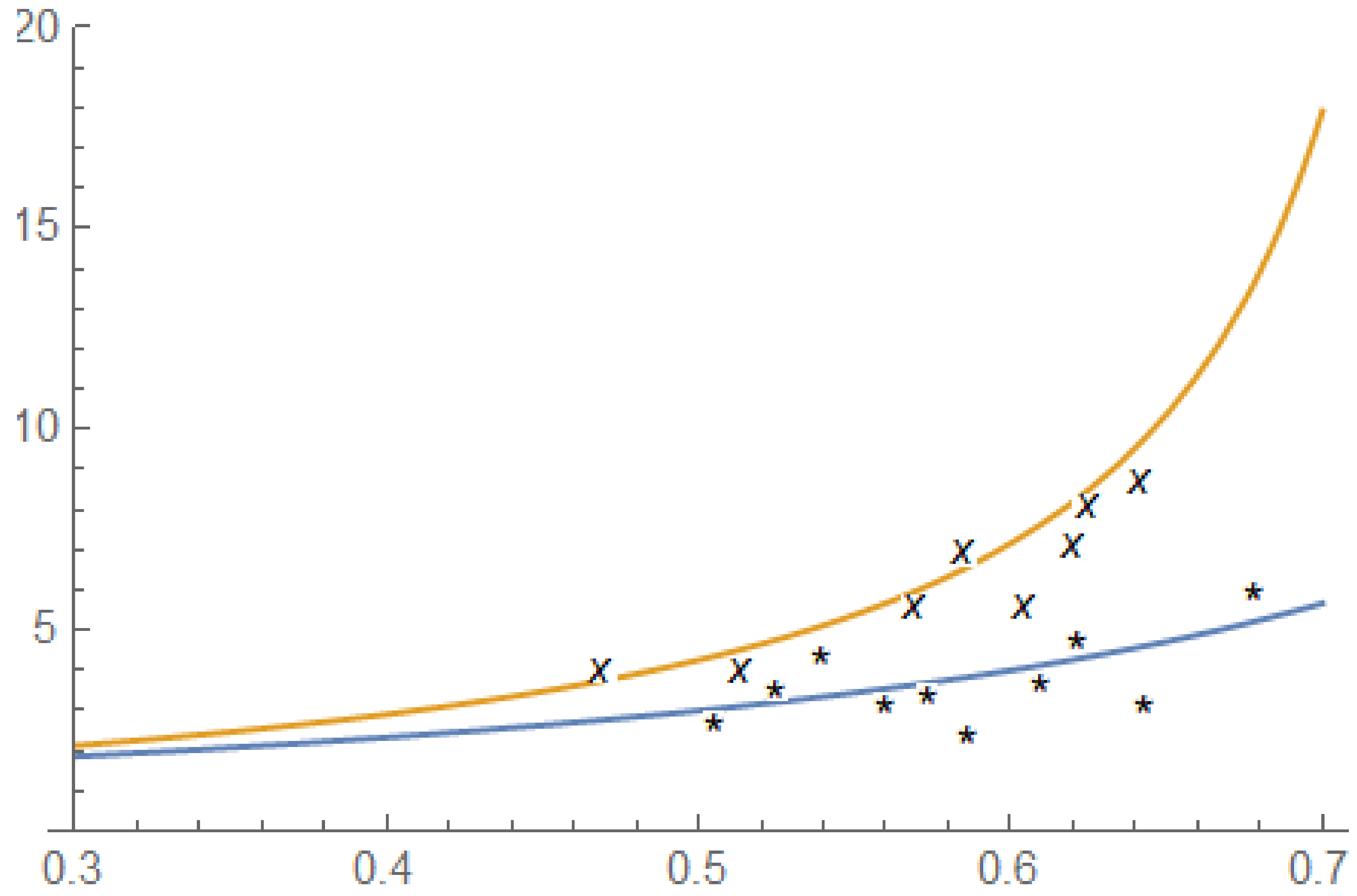
Where is the sleight of hand in these publications? In violation of asymptotic analysis rules.

MUCH ADO ABOUT NOTHING
EXPOSING TRICKS

How to get a „new model”. Example exposing tricks:

$$\frac{1+x}{1-x} \approx \frac{1+x}{1-x+\alpha x+\beta x^2} = \frac{1+x}{1-x-0.3x+0.01x^2} = \frac{1+x}{1-1.3x+0.01x^2},$$ where $\alpha x+\beta x^2$ is „something engineering”



Alchemy region

Conclusion.

Only the Riemann-Hilbert problem with proper asymptotic analysis can save the engineering world.

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Thank you for your attention

XI International Conference of the
Georgian Mathematical Union

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